Journal of Algebra, Number Theory: Advances and Applications
Volume 11, Number 2, 2014, Pages 65-77

COMPLEMENTARY TRIBONACCI SEQUENCES

DURSUN TASCI and SIMGE ODABAS

Department of Mathematics
Faculty of Science
University of Gazi
06500 Ankara
Turkey
e-mail: dtasci@gazi.edu
odabassmg@windowslive.com

Abstract

Assuming non-empty two sequences of positive integers are $a_1, a_2, \cdots, a_n$ and $c_1, c_2, \cdots, c_m$. If two sequences have no common elements and they are definitely increasing, they are called "complementary". When a_1 and a_2 are given, we obtain (a_i) by the help of (c_i), which occurs Tribonacci-like recurrence $a_i = c_{i-1} + c_{i-2} + c_{i-3}$ for $i \geq 4$. In this case, the sequence (c_i) is the complement of (a_i).

1. Introduction

For $a_1 \leq a_2$, the sequences (a_i) and (c_i) are defined by

$$a_i = c_{i-1} + c_{i-2} + c_{i-3} \text{ for } i \geq 4,$$

$$c_1 = \text{smallest number} \neq a_1, a_2, \qquad \text{(see [1])} \tag{1}$$

2010 Mathematics Subject Classification: Primary 11B20, 11B25.
Keywords and phrases: complementary equation, complementary sequences, arithmetic sequences.
Received April 10, 2014

c_i = smallest number $\neq a_1, a_2, \cdots, a_i, c_1, c_2, \cdots, c_{i-1}$ for $i \geq 2$. (see [1])

$a_3 = c_1 + c_2$.

Since $a_i > c_{i-1}$ and the sequence (c_i) is definitely increasing, we obtain $a_{i+1} - a_i = c_i - c_{i-3} \geq 0$, so the sequence (a_i) is strictly increasing for at least $i \geq 4$. By example,

$$(a_i) = (1, 4, 5, 11, 16, 21, 24, \cdots),$$

$$(c_i) = (2, 3, 6, 7, 8, 9, \cdots).$$

2. Results for $a_1 \equiv a_2 \equiv 3 \pmod 4$

In this case, the sequences (a_i) convert to arithmetic sequences whose difference are 4.

Theorem 2.1. *For* $a_1 \equiv a_2 \equiv 3 \pmod 4$, *we have*

$$a_i = 4i - 9 \textit{ for } i \geq 4 \textit{ and}$$

$$c_i = \lfloor \frac{4i}{3} \rfloor \textit{ for } i \geq 2.$$

Proof. Definition (Gargano and Quintas, see [2]): Let $f(n)$ be non-decreasing function, which maps the set of non-negative integers into itself and let

$f^*(n) =$ the number of positive solutions k of the inequality $f(k) < n$

or equivalently

$f^*(n) =$ the largest integer k such that $f(k) < n$; 0 if no such k exists.

Theorem (Lambek, Moser and Shapiro, see [3], [4], [5]): If $f(n)$ is a non-decreasing function, which maps the set of non-negative integers into itself, then $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ complementary sequences, and conversely if two strictly increasing sequences $F(n)$ and $G(n)$ are complementary, then there exists a non-decreasing function $f(n)$ such that $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$.

Theorem (Gargano and Quintas, see [2]) (The formula for a complementary arithmetic sequence): If $F(n) = dn + a(d > 1;\ n \in \mathbb{Z};\ a, d \in Z$ are fixed$)$ defines an arithmetic sequence of integers, then $G(n) = [\ {}^{(dn - a - 1)}\!/\!{}_{(d-1)}\ ]$ defines its complementary sequence.

By using above theorems and definition, we have $a_k = f(k) + k$ $\Rightarrow f(k) = a_k - k = 4k - 9 - k = 3k - 9$. The domain is $\mathbb{Z}$ of $f(k)$ and also since $f^*(i) =$ the number of positive solutions k of the inequality $f(k) < i$, we obtain $3k - 9 < i$. Then $f^*(i) = \lfloor \frac{i + 9 - 1}{3} \rfloor = \lfloor \frac{i + 8}{3} \rfloor$ and $c_i = f^*(i) + i = \lfloor \frac{4i + 8}{3} \rfloor$ for $i \geq 4$. It is clear that $c_i = \lfloor \frac{4i}{3} \rfloor$ for $i \geq 2$. And we obtain

$$a_i = c_{i-1} + c_{i-2} + c_{i-3},$$

$$a_i = \left\lfloor \frac{4(i-1)}{3} \right\rfloor + \left\lfloor \frac{4(i-2)}{3} \right\rfloor + \left\lfloor \frac{4(i-3)}{3} \right\rfloor = 4i - 9 \text{ for } i \geq 4.$$

For $a_3 \geq a_2 \geq a_1$, it is obvious that $c_1 = 1$. Using the third equation of (1), we have $c_i \neq a_j$ for $j \geq 1$ since $c_i \not\equiv 3 (\text{mod } 4)$. For $i = 3n (n \in \mathbb{Z}^+)$, we have $c_i = c_{i-1} + 2$, for $i = 3n + 1 (n \in \mathbb{Z}^+)$ and $i = 3n + 2 (n \in \mathbb{Z}^+)$, we have $c_i = c_{i-1} + 4$. For $i = 3n + 1 (n \in \mathbb{Z}^+)$, we have $c_i = c_{i-3} + 4$ since $c_{i-3} + 3$ is an a_j for $j \leq i$ as $c_{i-3} + 3 \equiv 0 (\text{mod } 4)$ and as $a_i = 4i - 9 > \frac{4i - 13}{3} = c_{i-3}$ for $i \geq 4$.

3. Results for $a_1 \equiv a_2 \not\equiv 3(\text{mod } 4)$

For consecutive differences of the sequences (a_i), the differences $\Delta_i = a_{i+1} - a_i$ for $i \geq 4$ are not always 4. Differences change for the given a_1, a_2, and i indices.

Theorem 3.1. *For* $a_1 = a_2 = 4j + r \geq 6$, $r = 0$, $r = 1$ *or* $r = 2$, *we have* a_3 *and* $\Delta_i = 4$ *for* $i \geq 4$, *except for the indices*

$$i = f_5(n, v, j, r)$$

$$= (3j+1)9^n + 1 + (v-2)\left(\frac{(v+r-4)(v+r-3)9^n}{2} + \frac{9^n - 1}{4}\right),$$

for $v = 1, 2, 3$ *and* $n = 0, 1, 2, \cdots$, *where* $\Delta_i = \begin{cases} 5, & \text{if } i \neq 3j+2, \\ 6, & \text{if } i = 3j+2. \end{cases}$

For $r = 0$,

$$i = f_3(n, v, j, r)$$

$$= (3j+1)3^{2n+1} + (v-1) + (v-2)\left(\frac{(v+r-4)(v+r-3)9^n}{2} + \frac{9^n - 1}{4}\right),$$

for $v = 1, 2, 3$ *and* $n = 0, 1, 2, \cdots$, *where* $\Delta_i = 3$.

For $r = 1$ *and* $r = 2$,

$$i = f_3(n, v, j, r)$$

$$= (3j+1)3^{2n+1} + (v-2) + 3(v-2)\left(\frac{(v+r-4)(v+r-3)9^n}{2} + \frac{9^n - 1}{4}\right),$$

for $v = 1, 2, 3$ *and* $n = 0, 1, 2, \cdots$, *where* $\Delta_i = 3$.

Proof. For $a_1 = a_2 \geq 6$, we obtain $c_1 = 1, c_2 = 2$ and $a_3 = 3$ by using (1). Since the sequence (c_i) starts as Theorem 3.1, we find $\Delta_i = 4$ for $4 \leq i < 3j + r$ from (1) and Theorem 3.1.

$$4 \leq i < 3j + r \Rightarrow 16 \leq 4i < 12j + 4r$$

$$\Rightarrow \frac{16}{3} \leq \frac{4i}{3} < \frac{12j + 4r}{3}$$

$$\Rightarrow \lfloor\frac{16}{3}\rfloor \leq \lfloor\frac{4i}{3}\rfloor < \lfloor\frac{12j + 4r}{3}\rfloor$$

$$\Rightarrow 5 \leq \left\lfloor\frac{4i}{3}\right\rfloor < 4j + r = a_1 = a_2,$$

$i = 3j + r$ determines the first c_i, which is different from the corresponding value in Theorem 2.1. For $a_1 = a_2 \geq 6$ and $i = 3j + r - 1$, $3j + r$, $3j + r + 1$, the values of c_i, a_i, and Δ_i are as in Tables 1, 2, 3 for the cases $r = 0$, $r = 1$, and $r = 2$.

For $r = 0$ and $q \in \mathbb{Z}^+ \cup \{0\}$, the values of (c_i) are as

$$(c_i) = \begin{cases} \left\lfloor\frac{4i}{3}\right\rfloor : 4 \leq i < 3j + r, \\ \left\lfloor\frac{4i}{3}\right\rfloor + 1 : 3j + r \leq i \leq 3j + (r + 1), \\ \left\lfloor\frac{4i}{3}\right\rfloor + 2 : 3j + (r + 1) < i \leq 3j + (r + 2), \\ \vdots \\ \vdots \\ \vdots \\ \left\lfloor\frac{4i}{3}\right\rfloor + 1 : 3j + r + (2q) \leq i \leq 3j + r + (2q + 1), \\ \left\lfloor\frac{4i}{3}\right\rfloor + 2 : 3j + r + (2q + 1) < i \leq 3j + r + (2q + 2). \end{cases}$$

It is clear that

for $i = 3j + 2,\ \Delta_i = 6,$

for $i \neq 3j + 2,\ \Delta_i = 5,$

for $4 \leq i < 3j,\ \Delta_i = 4.$

Table 1. The case $r = 0$

i	c_i	a_i	Δ_i
$3j - 1$	$4j - 2$	$12j - 13$	4
$3j$	$4j + 1$	$12j - 9$	5
$3j + 1$	$4j + 2$	$12j - 4$	5
$3j + 2$	$4j + 4$	$12j + 1$	6

For $r = 1$ and $q \in \mathbb{Z}^+ \cup \{0\}$, the values of (c_i) are as

$$(c_i) = \begin{cases} \left\lfloor \dfrac{4i}{3} \right\rfloor : 4 \leq i < 3j + r, \\ \left\lfloor \dfrac{4i}{3} \right\rfloor + 1 : 3j + r \leq i < 3j + (r + 1), \\ \left\lfloor \dfrac{4i}{3} \right\rfloor + 2 : 3j + (r + 1) \leq i < 3j + (r + 2), \\ \vdots \\ \vdots \\ \vdots \\ \left\lfloor \dfrac{4i}{3} \right\rfloor + 1 : 3j + r + (2q) \leq i < 3j + r + (2q + 1), \\ \left\lfloor \dfrac{4i}{3} \right\rfloor + 2 : 3j + r + (2q + 1) \leq i < 3j + r + (2q + 2). \end{cases}$$

It is clear that

for $i = 3j + 2,\ \Delta_i = 6,$

for $i \neq 3j + 2,\ \Delta_i = 5,$

for $4 \leq i < 3j + 1,\ \Delta_i = 4.$

Table 2. The case $r = 1$

i	c_i	a_i	Δ_i
$3j-1$	$4j-2$	$12j-13$	4
$3j$	$4j$	$12j-9$	4
$3j+1$	$4j+2$	$12j-5$	5
$3j+2$	$4j+4$	$12j$	6

For $r = 2$ and $q \in \mathbb{Z}^+ \cup \{0\}$, the values of (c_i) are as

$$(c_i) = \begin{cases} \left\lfloor \frac{4i}{3} \right\rfloor : 4 \le i < 3j + r, \\ \left\lfloor \frac{4i}{3} \right\rfloor + 2 : 3j + r \le i < 3j + (r+1), \\ \left\lfloor \frac{4i}{3} \right\rfloor + 1 : 3j + (r+1) \le i < 3j + (r+2), \\ \vdots \\ \vdots \\ \vdots \\ \left\lfloor \frac{4i}{3} \right\rfloor + 2 : 3j + r + (2q) \le i < 3j + r + (2q+1), \\ \left\lfloor \frac{4i}{3} \right\rfloor + 1 : 3j + r + (2q+1) \le i < 3j + r + (2q+2). \end{cases}$$

It is clear that

for $i = 3j + 2$, $\Delta_i = 6$,

for $i \neq 3j + 2$, $\Delta_i = 5$,

for $4 \le i < 3j + 2$, $\Delta_i = 4$.

Table 3. The case $r = 2$

i	c_i	a_i	Δ_i
$3j - 1$	$4j - 2$	$12j - 13$	4
$3j$	$4j$	$12j - 9$	4
$3j + 1$	$4j + 1$	$12j - 5$	4
$3j + 2$	$4j + 4$	$12j - 1$	6
$3j + 3$	$4j + 5$	$12j + 5$	5

So these occur exceptional differences $\Delta_i = 5$ or $\Delta_i = 6$ as shown for $n = 0$ and $v = 1, 2, 3.$ There exists three cases for $i = 3j + 2$ relating to differences $\Delta_i = 5$ or $\Delta_i = 6$.

$$\begin{aligned} 3j + 2 &= f_5(0, 2, j, 0) = f_5(0, 3, j, 0), \\ 3j + 2 &= f_5(0, 2, j, 1) = f_5(0, 3, j, 1), \\ 3j + 2 &= f_5(0, 1, j, 2) = f_5(0, 2, j, 2). \end{aligned} \tag{2}$$

We will see that differences $\Delta_x = 4, 5,$ and 6 in (a_i) determine $\Delta_x - 1$ consecutive numbers in (c_i), which make consecutive differences Δ_i which is 2 or 3. Differences $\Delta_i = 3$ come from only $\Delta_x = 5$ and 6 and make $\Delta_j = 5$ each. These cases (Δ_i) exactly.

Table 4. Differences 4 determined by $\Delta_x = 4$

i	c_i	a_i	Δ_i
x		a_x	4
		$a_x + 4$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$a_x - 1$		
	$a_x + 1$		
	$a_x + 2$	$2a_x + 5$	4
	$a_x + 3$	$2a_x + 9$	4
		$2a_x + 13$	

We assume that $\Delta_x = 5$ and $a_x = 4x - d_x$. We obtain exceptional differences $\Delta_y = 3$ and $\Delta_z = 5$ by using (1) as shown in Table 5. Additionally, Table 5 shows that if these indices are between x and y, another differences in (a_i) are 5 or 3, if these indices are between y and z, another differences in (a_i) are 3 or 5. With $a_i = 4i - d_i$, we have

$$d_{i+1} = d_i + 4 - \Delta_i. \tag{3}$$

In Table 5, $d_z = d_x$.

$a_z = 4z - d_z = 4z - d_x = 36x - 9d_x + 20$ determines $z = 9x - 2d_x + 5$.

From $a_y = 4y - d_y = 12x - 3d_x + 6$, we obtain $y = 3x + \frac{3}{2} - \left(\frac{3d_x - d_y}{4}\right)$.

Table 5. Differences 3, 4, and 5 determined by $\Delta_x = 5$

i	a_i	Δ_i
x	$4x - d_x$	5
$x+1$	$4x - d_x + 5$	
$\vdots$	$\vdots$	$\vdots$
$y = 3x + \frac{3}{2} - \left(\frac{3d_x - d_y}{4}\right)$	$12x - 3d_x + 6$	3
	$12x - 3d_x + 9$	4
	$12x - 3d_x + 13$	
$\vdots$	$\vdots$	$\vdots$
$z = 9x - 2d_x + 5$	$36x - 9d_x + 20$	5
	$36x - 9d_x + 25$	

If $\Delta_x = 6$, $\Delta_y = \Delta_{y+1} = 3$, and $\Delta_z = \Delta_{z+1} = 5$ are shown in Table 6 related to Table 5.

Table 6. Differences 3, 4, and 5 determined by $\Delta_x = 6$

i	a_i	Δ_i
x	$4x - d_x$	6
$x+1$	$4x - d_x + 6$	
$\vdots$	$\vdots$	$\vdots$
$y = 3x + \frac{3}{2} - \left(\frac{3d_x - d_y}{4}\right)$	$12x - 3d_x + 6$	3
$y+1$	$12x - 3d_x + 9$	3
	$12x - 3d_x + 12$	
$\vdots$	$\vdots$	$\vdots$
$z = 9x - 2d_x + 5$	$36x - 9d_x + 20$	5
$z+1$	$36x - 9d_x + 25$	5
	$36x - 9d_x + 30$	

We will observe that differences $\Delta_i = 3, 5$ or 6, which are seen in the first rows of Tables 7, 8, and 9 for $r = 0$, $r = 1$, and $r = 2$ with the help of Tables 4, 5, and 6 related to Tables 1, 2, and 3.

Table 7. Exceptional differences in the case $r = 0$

Δ_i	5	6	3	3	3	$\cdots$
n	0	0	0	0	0	$\cdots$
v	1	2, 3	1	2	3	$\cdots$
d_i	9	7	5	6	7	$\cdots$

Table 8. Exceptional differences in the case $r = 1$

Δ_i	5	6	3	3	3	$\cdots$
n	0	0	0	0	0	$\cdots$
v	1	2, 3	1	2	3	$\cdots$
d_i	9	8	5	6	7	$\cdots$

Table 9. Exceptional differences in the case $r = 2$

Δ_i	5	6	3	3	3	$\cdots$
n	0	0	0	0	0	$\cdots$
v	1, 2	3	1	2	3	$\cdots$
d_i	9	7	5	6	7	$\cdots$

Depending on (3) following values change only for $\Delta_i \neq 4$. Briefly

$$d_i = 10 - v \text{ for } \Delta_i = 5 \text{ and}$$

$$d_i = v + 4 \text{ for } \Delta_i = 3. \tag{4}$$

In case of $r = 0$, $r = 1$, and $r = 2$ for $\Delta_i = 5$, we have

$$d_{3j+r} = d_{3j} = 9, \quad \text{if } r = 0,$$

$$d_{3j+r} = d_{3j+1} = 9, \quad \text{if } r = 1,$$

$$d_{3j+r} = d_{3j+2} = 9, \quad \text{if } r = 2.$$

From Theorem 3.1 and Table 5, we know $x = f_5(n, v, j, r)$ and $z = 9x - 2d_x + 5$ for $\Delta_x = 5$ and $d_x = 10 - v$ from (3), we obtain

$$\begin{aligned} z &= 9x - 2d_x + 5 \\ &= 9f_5(n, v, j, r) - 2(10 - v) + 5 \\ &= 9f_5(n, v, j, r) + 2v - 15 = f_5(n+1, v, j, r). \end{aligned}$$

From (3), we know $d_y = v + 4$ and $y = 3x + \frac{3}{2} - \left(\frac{3d_x - d_y}{4}\right)$ from Table 5, we obtain

$$\begin{aligned} y &= f_3(n, v, j, r) \\ &= 3f_3(n, v, j, r) + \frac{3}{2} - \left(\frac{3(10 - v) - (v - 4)}{4}\right) \\ &= 3f_3(n, v, j, r) + v - 5. \end{aligned}$$

Corollary 3.1. *For* $a_1 = a_2 = 4j + r \geq 6$, *and* $r = 0, 1, 2$ *with* f_3 *and* f_5 *from Theorem* 3.1, *we have*

$$a_i = 4i - 9 \text{ for } 4 \leq i < 3j + r = f_5(0, 1, j, r),$$

and

$$a_i = 4i - d_m \text{ for } s < i \leq m,$$

where $\Delta_s \neq 4$ *and* $\Delta_m \neq 4$ *are two consecutive exceptional differences and*

$$d_m = \begin{cases} 10 - v, & \text{if } m = f_5(n, v, j, r), \\ v + 4, & \text{if } m = f_3(n, v, j, r). \end{cases}$$

4. Some Cases

For $a_1 = a_2 = 1, 2, 4,$ and 5, the sequences (a_i) are

(i) For $a_1 = a_2 = 1,$ we have $a_i = 4i - 7$ for $i \geq 4.$

(ii) For $a_1 = a_2 = 2,$ it exists cases as Table 5 for $i \geq 3.$

(iii) For $a_1 = a_2 = 4$ and $a_1 = a_2 = 5,$ it exists cases as Table 6 for $i \geq 4.$

References

[1] P. Jens-Bode, H. Harborth and C. Kimberling, Complementary Fibonacci sequences, Fibonacci Quart. 45 (2007), 254-264.

[2] M. L. Gargano and L. V. Quintas, Complementary arithmetic sequences, Congr. Numer. 192 (2003), 33-42.

[3] J. Lambek and L. Moser, Inverse and complementary sequences of natural numbers, The American Mathematical Monthly 61 (1954), 454-458.

[4] R. P. Grimaldi, Discrete and Combinatorial Mathematics, 4th Edition, Addison-Wesley, Redwood City, California, 1999.

[5] R. Honsberger, Inqenuity in mathematics, New Mathematics Library # 23, MAA (1970), 93-110.

■